Linear Algebra I

08/04/2021, Wednesday, 18:45 – 21:45

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1 Systems of linear equations (5 + 1 + 6 + 3 = 15 pts)

In this problem, we want to determine all polynomials $p(x) = a + bx + cx^2$ such that

$$\int_0^1 p(x)\,dx = 0 \quad \text{and} \quad \int_0^1 xp(x)\,dx = 1. \qquad (\star)$$

(a) Find linear equations in the unknowns a, b, c by using $(\star)$.

(b) Find the augmented matrix corresponding to the obtained linear equations.

(c) Put the augmented matrix into reduced row echelon form.

(d) Find the solution set using the obtained echelon form.

2 Determinants (15 pts)

Find $\alpha \in \mathbb{R}$ such that $\det\left(\begin{bmatrix} X+Y & X-Y \\ X-Y & X+Y \end{bmatrix}\right) = \alpha \det(X)\det(Y)$ for all matrices $X, Y \in \mathbb{R}^{n\times n}$.

3 Partitioned matrices and nonsingularity (5 + 10 = 15 pts)

Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix.

(a) Show that the matrix $M = \begin{bmatrix} A & A^{-1} \\ A^{-1} & A \end{bmatrix}$ is nonsingular if and only if A does not have an eigenvalue λ such that $\lambda^4 = 1$.

(b) Suppose that A does not have an eigenvalue λ such that $\lambda^4 = 1$. Find the inverse of M.

4 Vector spaces and linear transformations (3 + 3 + 3 + 1 + 5 = 15 pts)

For $A \in \mathbb{R}^{n\times n}$, let $S_A = \{X \in \mathbb{R}^{n\times n} \mid AX - XA = 0_{n,n}\}$ and $L_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be given by $L_A(X) = AX - XA$.

(a) Show that S_A is a subspace of $\mathbb{R}^{n\times n}$.

(b) Take $n = 2$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Determine the dimension of S_B.

(c) Show that L_A is a linear transformation.

(d) What is the relationship between S_A and L_A?

(e) Take $n = 2$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find the matrix representation of L_B relative to the bases

$$E = F = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

5 Least squares problem (15 pts)

Find the closest vector to

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

within the subspace

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

6 Eigenvalues and diagonalization $(5 + 5 + 5 = 15$ pts$)$

Let $n \geqslant 2$. Consider the matrix $A \in \mathbb{R}^{n \times n}$ given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

(a) Show that λ is an eigenvalue of A if and only if $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$.

(b) Suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Find a nonsingular matrix X such that $X^{-1}AX$ is diagonal.

(c) To see what happens if A has eigenvalues with higher multiplicities, consider the matrix

$$\begin{bmatrix} 0 & 1 \\ -\lambda^2 & 2\lambda \end{bmatrix}.$$

Find its eigenvalues and check if it is diagonalizable.

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10 pts free